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ON VAGUE BI-IDEALS AND VAGUE WEAKLY COMPLETELY PRIME IDEALS IN Γ-SEMIGROUPS

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Abstract. In this paper, the study of vague structures of Γ-semigroups is initiated by introducing the concepts of vague subsemigroup, vague bi-ideal and vague $(1,2)$-ideal in a Γ-semigroup and some fundamental results are obtained. Then the concept of vague weakly completely prime ideals in Γ-semigroups has been introduced and studied directly and via operator semigroups of a Γ-semigroup.

Keywords: Γ-semigroup, vague characteristic set, vague subsemigroup, vague bi-ideal, vague $(1,2)$-ideal, operator semigroups, Γ-semigroup.

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1. Introduction

In 1993, Gau and Buehrer [10] proposed the theory of vague sets as an improvement of the theory of fuzzy sets in approximating the real life situation. Vague sets are higher order fuzzy sets. A vague set A in the universe of discourse U is a pair (t_A, f_A) where t_A and f_A are fuzzy subsets of U satisfying the condition $t_A(u) \leq 1 - f_A(u)$ for all $u \in U$. Biswas [1] initiated the study of vague algebra by introducing the concepts of vague groups, vague normal groups. Khan, Ahmad and Biswas [17] introduced the notion of vague relations and studied some properties of them. Ramakrishna [20] continued this study by studying vague cosets, vague products and several properties related to them. In 2008, Jun and Park [13] introduced the notion of vague ideals in substraction algebra. Eswarlal [9] had introduced the concepts of vague ideals and normal vague ideals in semirings in 2008.

In 1986 the concept of Γ-semigroups was introduced by Sen and Saha [24] as a generalization of semigroups and ternary semigroups. Since then Γ-semigroups have been analyzed by lot of mathematicians, such as Chattopadhay [2, 3], Dutta and Adhikari [5, 6], Hila [11, 12], Chinram [4], Saha [21], Seth [25]. In 1965, after the introduction of fuzzy sets by Zadeh [26], reconsideration of the concept of classical mathematics began. As an immediate consequence fuzzy algebra is an well established branch of mathematics at present. Many authors have studied semigroups in terms of fuzzy sets. Kuroki [14, 15, 16] is the pioneer of this study. Motivated by Kuroki [14, 15, 16] and others S.K Majumder et. all. have studied Γ-semigroups in terms of fuzzy sets [7, 8, 19, 22, 23]. In a similar fashion after the commencement of the notion of vague sets by Gau and Buehrer [10] in 1993, many authors are presently trying to apply this concept in different algebraic structures. In this paper, we introduce the concepts of vague subsemigroup, vague

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bi-ideal, vague $(1,2)$-ideal and vague weakly completely prime ideal in Γ-semigroups and obtain some of their important properties. Various relationships between vague weakly completely prime ideals of a Γ-semigroup and vague weakly completely prime ideals (vague subsemigroups) of its operator semigroups have been obtained. An inclusion preserving bijection between the set of all vague subsemigroups of a Γ-semigroup and that of its operator semigroups is obtained by using the inclusion preserving bijection between their set of respective vague weakly completely prime ideals.

2. Basic definitions

Throughout this paper S denotes a Γ-semigroup unless or otherwise mentioned. At the first, we recall some basic definitions related to the theory of Γ-semigroups.

Let $S = \{x, y, z, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two non-empty sets. Then S is called a *Γ-semigroup* [24] if there exist a mapping $S \times \Gamma \times S \to S$(images to be denoted by $a\alpha b$) satisfying

(1) $x\gamma y \in S$,
(2) $(x\beta y)\gamma z = x\beta(y\gamma z)$, for all $x, y, z \in S$ and for all $\beta, \gamma \in \Gamma$.

A semigroup S is called *regular* [6] if for each element a of S, there exist an element $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Example 2.1. *Let $\Gamma = \{5, 7\}$. For any $x, y \in N$ and $\gamma \in \Gamma$, define $x\gamma y = x \cdot \gamma \cdot y$, where sign "$\cdot$" is the usual multiplication on N. Then N is a Γ-semigroup.*

Example 2.2. *Let S be the set of all negative rational numbers. Let $\Gamma = \{-\frac{1}{p} : p \text{ is prime}\}$. Let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Now, if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence, S is a Γ-semigroup.*

Remark 2.1. *The Γ-semigroup introduced by Sen and Saha [24] may be called* one sided Γ-semigroup. *Later Dutta and Adhikari [5] introduced both sided Γ-semigroup, where the operation $\Gamma \times S \times \Gamma$ to Γ was also taken into consideration. They defined operator semigroups for such Γ-semigroups.*

Example 2.3. *Let S be the set of all integers of the form $4n + 1$ and Γ be the set of all integers of the form $4n + 3$, where n is an integer. If $a\alpha b$ is $a + \alpha + b$ and $\alpha a\beta$ is $\alpha + a + \beta$(usual sum of integers) for all $a, b \in S$ and $\alpha, \beta \in \Gamma$. Then S is a both sided Γ-semigroup.*

A non-empty subset A of a Γ-semigroup S is called a *subsemigroup* of S if $A\Gamma A \subseteq A$. By a *left* (*right*) *ideal* of S we mean a non-empty subset A of S such that $S\Gamma A \subseteq A(A\Gamma S \subseteq A)$. By a *two sided ideal* or *simply an ideal*, we mean a non-empty subset A of S which is both a left and a right ideal of S. An ideal P of S is said to be *crisp prime* if, for any two crisp ideals A and B of $S, A\Gamma B \subseteq P$ implies that either $A \subseteq P$ or $B \subseteq P$.

A subsemigroup A of a Γ-semigroup S is called a *bi-ideal* of S if $A\Gamma S\Gamma A \subseteq A$. A subsemigroup A of a Γ-semigroup S is called a $(1,2)$-*ideal* [22] of S if $A\Gamma S\Gamma A\Gamma A \subseteq A$.

Now, we recall some definitions related to fuzzy sets in Γ-semigroups, for more results see [22].

- A function μ from a non-empty set S to the unit interval $[0, 1]$ is called a *fuzzy subset* of S.
- A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy subsemigroup* of S, if $\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\},\ \forall x, y \in S,\ \forall \gamma \in \Gamma$.
- A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy left ideal* of S, if $\mu(x\gamma y) \geq \mu(y),\ \forall x, y \in S,\ \forall \gamma \in \Gamma$.

- A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy right ideal* of S, if $\mu(x\gamma y) \geq \mu(x), \ \forall x, y \in S, \ \forall \gamma \in \Gamma$.
- A non-empty fuzzy subset μ of a Γ-semigroup S is called a *fuzzy ideal* of S, if μ is a fuzzy left ideal and a fuzzy right ideal of S.
- A fuzzy ideal μ of a Γ-semigroup S is called a *fuzzy weakly completely prime ideal* of S, if $\mu(x) \geq \mu(x\gamma y)$ or $\mu(y) \geq \mu(x\gamma y), \ \forall x, y \in S, \ \forall \gamma \in \Gamma$.
- A fuzzy subsemigroup μ of a Γ-semigroup S is called a *fuzzy bi-ideal* of S, if $\mu(x\alpha y\beta z) \geq$ $\geq \min\{\mu(x), \mu(z)\}, \ \forall x, y, z \in S, \ \forall \alpha, \beta \in \Gamma$.
- A fuzzy subsemigroup μ of a Γ-semigroup S is called a *fuzzy* $(1,2)$*-ideal* of S, if $\mu(x\alpha w\beta(y\gamma z)) \geq \min\{\mu(x), \mu(y), \mu(z)\}, \ \forall x, w, y, z \in S, \ \forall \alpha, \beta, \gamma \in \Gamma$.

Now, we discuss the following concepts regarding vague sets, see [9].

Let $S = \{x_1, x_2, \ldots, x_n\}$ be the universe of discourse. The membership function for fuzzy sets can take values from the closed interval $[0,1]$. A fuzzy set A in S is defined as the set of ordered pairs $A = \{(x, \mu_A(x)) : x \in S\}$, where $\mu_A(x)$ is the grade of membership of the element x in the set A. The truth of the statement *the element x belongs to the set A* increases as the value of $\mu_A(x)$ closes to 1. Gau and Buehrer [10] noticed that this single value of $\mu_A(x)$ combines the *evidence for x* and the *evidence against x*. It does not indicate the evidence for x and the evidence against x, and it does not also indicate how much there is of each. The necessity to introduce the concept of vague sets was originated from this point. Vague sets are different kind of fuzzy sets, which could be treated as a generalization of Zadeh's fuzzy sets [26].

- A *vague set* A in the universe of discourse S is a pair (t_A, f_A), where $t_A : S \to [0,1]$ and $f_A : S \to [0,1]$ are mappings (called truth membership function and false membership function respectively), where $t_A(x)$ is a lower bound of the grade of membership of x derived from the *evidence for* x and $f_A(x)$ is a lower bound on the negation of x derived from the *evidence against* x and $t_A(x) + f_A(x) \leq 1 \forall x \in S$.
- The interval $[t_A(x), 1 - f_A(x)]$ is called the *vague value* of x in A, and it is denoted by $V_A(x)$, i.e., $V_A(x) = [t_A(x), 1 - f_A(x)]$.
- A vague set A of S is said to be contained in another vague set B of S, i.e., $A \subseteq B$, if and only if $V_A(x) \leq V_B(x)$, i.e., $t_A(x) \leq t_B(x)$ and $1 - f_A(x) \leq 1 - f_B(x), \ \forall x \in S$.
- Two vague sets A and B of S are equal, i.e., $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$, i.e., $V_A(x) \leq V_B(x)$ and $V_B(x) \leq V_A(x), \ \forall x \in S$, which implies that $t_A(x) = t_B(x)$ and $1 - f_A(x) = 1 - f_B(x)$.
- The *union* of two vague sets A and B of S with respective truth membership and false membership functions t_A, f_A and t_B, f_B is a vague set C of S, written as $C = A \cup B$, whose truth membership and false membership functions are related to those of A and B by $t_C = \max(t_A, t_B)$ and $1 - f_C = \max(1 - f_A, 1 - f_B) = 1 - \min(f_A, f_B)$.
- The *intersection* of two vague sets A and B of S with respective truth membership and false membership functions t_A, f_A and t_B, f_B is a vague set C of S, written as $C = A \cap B$, whose truth membership and false membership functions are related to those of A and B by $t_C = \min(t_A, t_B)$ and $1 - f_C = \min(1 - f_A, 1 - f_B) = 1 - \max(f_A, f_B)$.
- A vague set A of S with $t_A(x) = 0$ and $f_A(x) = 1, \ \forall x \in S$, is called the *zero vague set* of S.
- A vague set A of S with $t_A(x) = 1$ and $f_A(x) = 0, \ \forall x \in S$, is called the *unit vague set* of S.
- Let A be a vague set of the universe S with truth membership function t_A and false membership function f_A. For $\alpha, \beta \in [0,1]$ with $\alpha \leq \beta$, the (α, β)*-cut or vague cut* [9] of

the vague set A is a crisp subset $A_{(\alpha,\beta)}$ of S given by $A_{(\alpha,\beta)} = \{x \in S : V_A(x) \geq (\alpha, \beta)\}$, *i.e.*, $A_{(\alpha,\beta)} = \{x \in S : t_A(x) \geq \alpha \text{ and } 1 - f_A(x) \geq \beta\}$.

- Let $I[0,1]$ denotes the family of all closed sub intervals of $[0,1]$. $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ are two elements of $I[0,1]$. We call $I_1 \geq I_2$, if $a_1 \geq a_2$ and $b_1 \geq b_2$ with the order in $I[0,1]$ is a lattice with operations min. or inf and max. or sup given by $imin\{I_1, I_2\} = [\min\{a_1, a_2\}, \min\{b_1, b_2\}]$ and $imax\{I_1, I_2\} = [\max\{a_1, a_2\}, \max\{b_1, b_2\}]$. Also we denote $I_1 + I_2 = [a_1 + a_2, b_1 + b_2]$ [9].

The *α-cut*, A_α of the vague set A is the (α, α)-cut of A and hence it is given by $A_\alpha = \{x \in S : t_A(x) \geq \alpha\}$.

Let $\delta = (t_\delta, f_\delta)$ be a vague set of a Γ-semigroup S. For any subset T of S, the characteristic function of T taking values in $[0,1]$ is a vague set $\delta_T = (t_{\delta_T}, f_{\delta_T})$ given by

$$V_{\delta_T}(x) = \begin{cases} 1, & \text{if } x \in T, \\ 0, & \text{otherwise,} \end{cases}$$

i.e.,

$$t_{\delta_T}(x) = \begin{cases} 1, & \text{if } x \in T, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad f_{\delta_T}(x) = \begin{cases} 0, & \text{if } x \in T, \\ 1, & \text{otherwise.} \end{cases}$$

Then δ_T is called the *vague characteristic set* of T in $[0,1]$.

Let S be a Γ-semigroup. Then [18]:

- A non-empty vague set A of S is called a *left vague ideal* of S, if $V_A(x\gamma y) \geq V_A(y)$, i.e., $t_A(x\gamma y) \geq t_A(y)$ and $1 - f_A(x\gamma y) \geq 1 - f_A(y)$, $\forall x, y \in S$, $\forall \gamma \in \Gamma$.
- A non-empty vague set A of S is called a *right vague ideal* of S, if $V_A(x\gamma y) \geq V_A(x)$, i.e., $t_A(x\gamma y) \geq t_A(x)$ and $1 - f_A(x\gamma y) \geq 1 - f_A(x)$, $\forall x, y \in S$, $\forall \gamma \in \Gamma$.
- A non-empty vague set A of S is called a *vague ideal* of S, if it is a left vague ideal and a right vague ideal of S.

3. Vague subsemigroup, vague bi-ideal and vague $(1,2)$-ideal

Definition 3.1. *Let S be a Γ-semigroup. A non-empty vague set A of S is called a vague subsemigroup of S, if $V_A(x\gamma y) \geq \min\{V_A(x), V_A(y)\}$, i.e., $t_A(x\gamma y) \geq \min\{t_A(x), t_A(y)\}$ and $1 - f_A(x\gamma y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}$, $\forall x, y \in S$, $\forall \gamma \in \Gamma$.*

Definition 3.2. *Let S be a Γ-semigroup. A vague subsemigroup A of S is called a vague bi-ideal of S, if $V_A(x\alpha y\beta z) \geq \min\{V_A(x), V_A(z)\}$, i.e., $t_A(x\alpha y\beta z) \geq \min\{t_A(x), t_A(z)\}$ and $1 - f_A(x\alpha y\beta z) \geq \min\{1 - f_A(x), 1 - f_A(z)\}$, $\forall x, y, z \in S$, $\forall \alpha, \beta \in \Gamma$.*

Definition 3.3. *Let S be a Γ-semigroup. A vague subsemigroup A of S is called a vague $(1,2)$-ideal of S if $V_A(x\alpha w\beta(y\gamma z)) \geq \min\{V_A(x), V_A(y), V_A(z)\}$, i.e., $t_A(x\alpha w\beta(y\gamma z)) \geq \min\{t_A(x), t_A(y), t_A(z)\}$ and $1 - f_A(x\alpha w\beta(y\gamma z)) \geq \min\{1 - f_A(x), 1 - f_A(y), 1 - f_A(z)\}, \forall x, w, y, z \in S$, $\forall \alpha, \beta, \gamma \in \Gamma$.*

Example 3.1. *Let S be the set of all non-positive integers and Γ be the set of all non-positive even integers. Then S is a Γ-semigroup, if $a\gamma b$ and $\alpha a\beta$ denote the usual multiplication of integers a, γ, b and α, a, β respectively, where $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Let us define a vague set A of S as:*

$$t_A(x) = \begin{cases} 1, & \textit{if } x = 0, \\ 0.1, & \textit{if } x = -1, -2, \\ 0.2, & \textit{if } x < -2, \end{cases}$$

and

$$f_A(x) = \begin{cases} 0, & \text{if } x = 0, \\ 0.8, & \text{if } x = -1, -2, \\ 0.7, & \text{if } x < -2. \end{cases}$$

Then A is a vague subsemigroup and a vague bi-ideal of S.

Theorem 3.1. *A necessary and sufficient condition for a vague set A of a Γ-semigroup S to be a vague subsemigroup (vague bi-ideal) of S is that, t_A and $1 - f_A$ are fuzzy subsemigroups (resp. fuzzy bi-ideals) of S.*

Proof. Let A be a vague subsemigroup of S and $x, y \in S, \gamma \in \Gamma$. Then $V_A(x\gamma y) \geq \min\{V_A(x), V_A(y)\}$ which implies that $t_A(x\gamma y) \geq \min\{t_A(x), t_A(y)\}$ and $1 - f_A(x\gamma y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}$. Hence, t_A and $1 - f_A$ are fuzzy subsemigroups of S. The converse part of the theorem follows easily from the definition. For the case of bi-ideal, the proof is similar. □

Theorem 3.2. *Let S be a Γ-semigroup and A be a non-empty vague set of S. Then, A is a vague subsemigroup (vague bi-ideal) of S, if and only if the (α, β)-cut, $A_{(\alpha,\beta)}$ of A is a subsemigroup (respectively, bi-ideal) of S, for all $\alpha, \beta \in [0, 1]$.*

Proof. Let A be a vague subsemigroup of S and $\alpha, \beta \in [0, 1]$. Then there exists $z \in S$ such that $t_A(z) = \alpha, 1 - f_A(z) = \beta$ and so $z \in A_{(\alpha,\beta)}$. Thus, $A_{(\alpha,\beta)} \neq \phi$. Let $x, y \in A_{(\alpha,\beta)}, \gamma \in \Gamma$. Then $t_A(x), t_A(y) \geq \alpha$ and $1 - f_A(x), 1 - f_A(y) \geq \beta$. Since A is a vague subsemigroup S, so we have $t_A(x\gamma y) \geq \min\{t_A(x), t_A(y)\} \geq \alpha$ and $1 - f_A(x\gamma y) \geq \min\{1 - f_A(x), 1 - f_A(y)\} \geq \beta$. Then $x\gamma y \in A_{(\alpha,\beta)}$, i.e., $A_{(\alpha,\beta)}\Gamma A_{(\alpha,\beta)} \subseteq A_{(\alpha,\beta)}$. Thus, $A_{(\alpha,\beta)}$ is a subsemigroup of S.

Conversely, let $A_{(\alpha,\beta)}$ be a subsemigroup of S for all $\alpha, \beta \in [0, 1]$. Again, let $x, y \in S, \gamma \in \Gamma$. Let $t_A(x) = \alpha_1, t_A(y) = \alpha_2$ and $1 - f_A(x) = \beta_1, 1 - f_A(y) = \beta_2$. If $\alpha_1 \geq \alpha_2, \beta_1 \geq \beta_2$, then $x, y \in$ $\in A_{(\alpha_2,\beta_2)}$, which implies that $x\gamma y \in A_{(\alpha_2,\beta_2)}$ (by hypothesis). Then, $t_A(x\gamma y) \geq \alpha_2 = \min\{t_A(x), t_A(y)\}$ and $1 - f_A(x\gamma y) \geq \beta_2 = \min\{1 - f_A(x), 1 - f_A(y)\}$. Hence, A is a vague subsemigroup of S. Again, let $\alpha_2 > \alpha_1, \beta_2 > \beta_1$. Then $x, y \in A_{(\alpha_1,\beta_1)}$, which implies that $x\gamma y \in A_{(\alpha_1,\beta_1)}$(by hypothesis). Then $t_A(x\gamma y) \geq \alpha_1 = \min\{t_A(x), t_A(y)\}$ and $1 - f_A(x\gamma y) \geq \beta_1 = \min\{1 - f_A(x), 1 - f_A(y)\}$. Therefore, A is a vague subsemigroup of S. Similarly, we can prove the other case, too. □

Theorem 3.3. *Let T be a non-empty subset of a Γ-semigroup S and δ_T be a vague characteristic set of T. Then δ_T is a vague subsemigroup (respectively, vague bi-ideal) of S, if and only if T is a subsemigroup (respectively, bi-ideal) of S.*

Proof. Let δ_T be a vague subsemigroup of S. Let $x, y \in T, \gamma \in \Gamma$. Then we have $V_{\delta_T}(x) =$ $= V_{\delta_T}(y) = 1$. Since δ_T is a vague subsemigroup of S, so we have $V_{\delta_T}(x\gamma y) \geq \min\{V_{\delta_T}(x), V_{\delta_T}(y) =$ 1. But actually $V_{\delta_T}(x\gamma y) \leq 1$ whence $V_{\delta_T}(x\gamma y) = 1$. Consequently, $x\gamma y \in T$. Hence T is a crisp subsemigroup of S.

Conversely, let T be a crisp subsemigroup of S. Let $x, y \in S, \gamma \in \Gamma$ then $x\gamma y \in T$ if $x, y \in T$. It follows that $V_{\delta_T}(x\gamma y) = 1 = V_{\delta_T}(x) = V_{\delta_T}(y) = \min\{V_{\delta_T}(x), V_{\delta_T}(y)\}$. Then δ_T is a vague subsemigroup of S. Again let $x \notin I, y \notin I$. Then,

Case (1) : If $x\gamma y \notin I$, then $V_{\delta_T}(x\gamma y) \geq 0 = \min\{V_{\delta_T}(x), V_{\delta_T}(y)\}$.

Case (2) : If $x\gamma y \in I$, then $V_{\delta_T}(x\gamma y) = 1 \geq 0 = \min\{V_{\delta_T}(x), V_{\delta_T}(y)\}$. Hence δ_T is a vague subsemigroup of S. Similarly, we can prove the other case also. □

Proposition 3.1. *Every vague bi-ideal of a regular Γ-semigroup S is a vague subsemigroup of S.*

Proof. Let A be vague bi-ideal of S and S be regular. Let $x, y \in S, \gamma \in \Gamma$. Since S is regular, there exists $z \in S$ and $\alpha, \beta \in \Gamma$ such that $y = y\alpha z\beta y$. Then, we obtain

$$t_A(x\gamma y) = t_A(x\gamma(y\alpha z\beta y)) = t_A(x\gamma(y\alpha z)\beta y) \geq \min\{t_A(x), t_A(y)\}$$

and

$$1 - f_A(x\gamma y) = 1 - f_A(x\gamma(y\alpha z\beta y)) = 1 - f_A(x\gamma(y\alpha z)\beta y) \geq \min\{1 - f_A(x), 1 - f_A(y)\}.$$

Thus, A is a vague subsemigroup of S. □

Theorem 3.4. *In a regular* Γ*-semigroup* S*, the following conditions are equivalent:*

(1) *A is a vague bi-ideal of S,*
(2) *A is a vague* $(1, 2)$*-ideal of* S.

Proof. Let A be a vague bi-ideal of S and $x, w, y, z \in S, \alpha, \beta, \gamma \in \Gamma$. Then,

$$\begin{aligned} t_A(x\alpha w\beta(y\gamma z)) &= t_A((x\alpha w\beta y)\gamma z) \geq \\ &\geq \min\{t_A(x\alpha w\beta y), t_A(z)\} \geq \\ &\geq \min\{\min\{t_A(x), t_A(y)\}, t_A(z)\} = \\ &= \min\{t_A(x), t_A(y), t_A(z)\}. \end{aligned}$$

Also,

$$\begin{aligned} 1 - f_A(x\alpha w\beta(y\gamma z)) &= 1 - f_A((x\alpha w\beta y)\gamma z) \geq \\ &\geq \min\{1 - f_A(x\alpha w\beta y), 1 - f_A(z)\} \geq \\ &\geq \min\{\min\{1 - f_A(x), 1 - f_A(y)\}, 1 - f_A(z)\} = \\ &= \min\{1 - f_A(x), 1 - f_A(y), 1 - f_A(z)\}. \end{aligned}$$

Thus, A is a vague $(1, 2)$-ideal of S.

Conversely, let S be a regular Γ-semigroup and A be a vague $(1, 2)$-ideal of S. Let $x, w, y \in S$; $\alpha, \delta \in \Gamma$. Since S is regular, we have $x\alpha w \in x\Gamma S\Gamma x$, which implies that $x\alpha w = x\beta s\gamma x$ for some $s \in S; \beta, \gamma \in \Gamma$. Then,

$$\begin{aligned} t_A(x\alpha w\delta y) &= t_A((x\beta s\gamma x)\delta y) = \\ &= t_A(x\beta s\gamma(x\delta y)) \geq \\ &\geq \min\{t_A(x), t_A(x), t_A(y)\} = \\ &= \min\{t_A(x), t_A(y)\}. \end{aligned}$$

Also,

$$\begin{aligned} 1 - f_A(x\alpha w\delta y) &= 1 - f_A((x\beta s\gamma x)\delta y) = \\ &= 1 - f_A(x\beta s\gamma(x\delta y)) \geq \\ &\geq \min\{1 - f_A(x), 1 - f_A(x), 1 - f_A(y)\} = \\ &= \min\{1 - f_A(x), 1 - f_A(y)\}. \end{aligned}$$

Therefore, A is a vague bi-ideal of S. □

Definition 3.4. [20] *Let A and B be two vague sets of a* Γ*-semigroup S. Then the product of A and B, is denoted by $A \circ B$ and is defined by* $V_{A\circ B}(x) = \sup\{\min\{V_A(y), V_B(z)\} : x = y\gamma z\}$, *for* $y, z \in S, \gamma \in \Gamma$, *i.e.,* $t_{A\circ B}(x) = \sup\{\min\{t_A(y), t_B(z)\} : x = y\gamma z\}$ *and* $f_{A\circ B}(x) = = \inf\{\max\{f_A(y), f_B(z)\} : x = y\gamma z\}$, *for* $y, z \in S, \gamma \in \Gamma$.

Theorem 3.5. *A vague set A of a* Γ*-semigroup S is a vague subsemigroup of S, if and only if* $A \circ A \subseteq A$.

Proof. If $A \circ A \subseteq A$, then for $x, y \in S, \gamma \in \Gamma$ we have, $V_A(x\gamma y) \geq V_{A\circ A}(x\gamma y) \geq \min\{V_A(x), V_A(y)\}$. Hence, A is a vague subsemigroup of S.

Conversely, let A be a vague subsemigroup of S and for every $x \in S$ there exist $y, z \in S, \gamma \in \Gamma$ such that $x = y\gamma z$. Then $V_{A\circ A}(x) = \sup\{\min\{V_A(y), V_A(z)\} : x = y\gamma z\} \leq \min\{V_A(y), V_A(z)\} \leq$ $\leq V_A(y\gamma z) = V_A(x)$. In otherwise, $V_{A\circ A}(x) = 0 \leq V_A(x)$. Hence $A \circ A \subseteq A$. □

Theorem 3.6. *In a Γ-semigroup S the following conditions are equivalent:*

(1) *A is a vague bi-ideal of S,*
(2) *$A \circ A \subseteq A$ and $A \circ B \circ A \subseteq A$, where B is the vague characteristic set of S.*

Proof. Let us assume that (1) holds. Since A is a vague bi-ideal of S, so A is a vague subsemigroup of S. Then by Theorem 3.5, $A \circ A \subseteq A$. Let $a \in S$. Then, there exist $x, y, p, q \in S, \beta, \gamma \in \Gamma$ such that $a = x\gamma y$ and $x = p\beta q$. Since A is a vague bi-ideal of S, so we have $V_A(p\beta q\gamma y) \geq \min\{V_A(p), V_A(y)\}$. Then, we have

$$\begin{aligned} V_{A\circ B\circ A}(a) &= \sup\{\min\{V_{A\circ B}(x), V_A(y)\} : a = x\gamma y\} = \\ &= \sup\{\min\{\sup\{\min\{V_A(p), V_B(q)\} : x = p\beta q\}, V_A(y)\} : a = x\gamma y\} = \\ &= \sup\{\min\{\sup\{\min\{V_A(p), 1\} : x = p\beta q\}, V_A(y)\} : a = x\gamma y\} = \\ &= \sup\{\min\{V_A(p), V_B(y)\} : a = x\gamma y\} \leq \\ &\leq V_A(p\beta q\gamma y) = \\ &= V_A(a). \end{aligned}$$

In otherwise, $V_{A\circ B\circ A}(x) = 0 \leq V_A(x)$. Hence $A \circ B \circ A \subseteq A$.

Conversely, let us assume that (2) holds. Since $A \circ A \subseteq A$, so A is a vague subsemigroup of S. Let $x, y, z \in S, \beta, \gamma \in \Gamma$ and $a = x\beta y\gamma z$. Since $A \circ B \circ A \subseteq A$, we have

$$\begin{aligned} V_A(x\beta y\gamma z) &= V_A(a) \geq V_{A\circ B\circ A}(a) = \\ &= \sup\{\min\{V_{A\circ B}(x\beta y), V_A(z)\} : a = x\beta y\gamma z\} \geq \\ &\geq \min\{V_{A\circ B}(p), V_A(z)\} \text{ (where } p = x\beta y) = \\ &= \min\{\sup\{\min\{V_A(x), V_B(y)\} : p = x\beta y\}, V_A(z)\} \geq \\ &\geq \min\{\min\{V_A(x), 1\}, V_A(z)\} = \\ &= \min\{V_A(x), V_A(z)\}. \end{aligned}$$

Therefore, A is a vague bi-ideal of S. □

4. Vague weakly completely prime ideal

Definition 4.1. *A vague ideal A of a Γ-semigroup S is called a vague weakly completely prime ideal of S, if*

$$V_A(x) \geq V_A(x\gamma y) \quad \text{or} \quad V_A(y) \geq V_A(x\gamma y),$$

i.e.,

$$t_A(x) \geq t_A(x\gamma y) \quad \text{and} \quad 1 - f_A(x) \geq 1 - f_A(x\gamma y)$$

or

$$t_A(y) \geq t_A(x\gamma y) \quad \text{and} \quad 1 - f_A(y) \geq 1 - f_A(x\gamma y),$$

for all $x, y \in S$ and for all $\gamma \in \Gamma$.

Example 4.1. *Let S be the set of all non-positive integers and Γ be the set of all non-positive even integers. Then S is a Γ-semigroup if $a\gamma b$ and $\alpha a\beta$ denote the usual multiplication of* *integers a, γ, b and α, a, β, respectively, where $a, b \in S$ and $\alpha, \beta, \gamma \in \Gamma$. Let us define a vague set A of S as:*

$$t_A(x) = \begin{cases} 1, & \text{if } x = 0, -1, \\ 0.1, & \text{if } x < -1, \end{cases}$$

and

$$f_A(x) = \begin{cases} 0, & \text{if } x = 0, -1, \\ 0.8, & \text{if } x < -1. \end{cases}$$

Then A is a vague weakly completely prime ideal of S.

Theorem 4.1. *Let A be a non-empty vague set of a Γ-semigroup S. Then $1 - A$ is a vague subsemigroup of S if and only if A is a vague weakly completely prime ideal of S.*

Proof. Let $1 - A$ be a vague subsemigroup of S. Let $x, y \in S$ and $\gamma \in \Gamma$. Then, $1 - t_A(x\gamma y) \geq \geq \min\{1 - t_A(x), 1 - t_A(y)\} \Leftrightarrow 1 - t_A(x\gamma y) \geq 1 - \max\{t_A(x), t_A(y)\} \Leftrightarrow \max\{t_A(x), t_A(y)\} \geq \geq t_A(x\gamma y) \Leftrightarrow t_A(x) \geq t_A(x\gamma y)$ or $t_A(y) \geq t_A(x\gamma y)$ and $f_A(x\gamma y) \geq \min\{f_A(x), f_A(y)\} \Leftrightarrow \Leftrightarrow 1 - f_A(x\gamma y) \leq 1 - \min\{f_A(x), f_A(y)\} \Leftrightarrow \max\{1 - f_A(x), 1 - f_A(y)\} \geq 1 - f_A(x\gamma y) \Leftrightarrow \Leftrightarrow 1 - f_A(x) \geq 1 - f_A(x\gamma y)$ or $1 - f_A(y) \geq 1 - f_A(x\gamma y)$. Thus, A is a vague weakly completely prime ideal of S. □

Theorem 4.2. *A necessary and sufficient condition for a vague set A of a Γ-semigroup S to be a vague weakly completely prime ideal of S is that, t_A and $1 - f_A$ are fuzzy weakly completely prime ideals of S.*

Proof. Let A be a vague weakly completely prime ideal of S and $x, y \in S, \gamma \in \Gamma$. Then $V_A(x) \geq \geq V_A(x\gamma y)$ or $V_A(y) \geq V_A(x\gamma y)$ which implies that $t_A(x) \geq t_A(x\gamma y)$ and $1 - f_A(x) \geq 1 - f_A(x\gamma y)$ or $t_A(y) \geq t_A(x\gamma y)$ and $1 - f_A(y) \geq 1 - f_A(x\gamma y)$. Then $t_A(x) \geq t_A(x\gamma y)$ or $t_A(y) \geq t_A(x\gamma y)$ and $1 - f_A(x) \geq 1 - f_A(x\gamma y)$ or $1 - f_A(y) \geq 1 - f_A(x\gamma y)$. Hence t_A and $1 - f_A$ are fuzzy weakly completely prime ideals of S. The converse part of the theorem follows easily from the definition. □

Theorem 4.3. *Let S be a Γ-semigroup and A be a non-empty vague set of S. Then the following conditions are equivalent:*

(1) *A is a vague weakly completely prime ideal of S,*
(2) *for any $\alpha, \beta \in [0, 1], A_{(\alpha,\beta)}$ (if it is non-empty) is a prime ideal of S.*

Proof. Let A be a vague weakly completely prime ideal of S. Let $\alpha, \beta \in [0, 1]$ be such that $A_{(\alpha,\beta)}$ is non-empty. Let $x, y \in S, x\Gamma y \subseteq A_{(\alpha,\beta)}$. Then $t_A(x\gamma y) \geq \alpha$ and $1 - f_A(x\gamma y) \geq \beta$, $\forall \gamma \in \Gamma$. Since A is a vague weakly completely prime ideal of S, so we have $V_A(x) \geq V_A(x\gamma y)$ or $V_A(y) \geq V_A(x\gamma y)$, i.e., $t_A(x) \geq t_A(x\gamma y)$ and $1 - f_A(x) \geq 1 - f_A(x\gamma y)$ or $t_A(y) \geq t_A(x\gamma y)$ and $1 - f_A(y) \geq 1 - f_A(x\gamma y)$. Then, $t_A(x) \geq \alpha$ and $1 - f_A(x) \geq \beta$ or $t_A(y) \geq \alpha$ and $1 - f_A(y) \geq \beta$ which implies that $x \in A_{(\alpha,\beta)}$ or $y \in A_{(\alpha,\beta)}$. Therefore, $A_{(\alpha,\beta)}$ is a prime ideal of S.

Conversely, let us suppose that $A_{(\alpha,\beta)}$ is a prime ideal of S. Let $t_A(x\gamma y) = \alpha$ and $1 - f_A(x\gamma y) = = \beta$ (we note here that since $t_A(x\gamma y), 1 - f_A(x\gamma y) \in [0, 1]$, $\forall \gamma \in \Gamma, t_A(x\gamma y)$ and $1 - f_A(x\gamma y)$ exist). Then, $t_A(x\gamma y) \geq \alpha$ and $1 - f_A(x\gamma y) \geq \beta$, $\forall \gamma \in \Gamma$. Hence, $A_{(\alpha,\beta)}$ is non-empty and $x\Gamma y \subseteq A_{(\alpha,\beta)}$. Since $A_{(\alpha,\beta)}$ is a prime ideal of S, so we have $x \in A_{(\alpha,\beta)}$ or $y \in A_{(\alpha,\beta)}$. Then $t_A(x) \geq \alpha$ and $1 - f_A(x) \geq \beta$ or $t_A(y) \geq \alpha$ and $1 - f_A(y) \geq \beta$ which implies that $t_A(x) \geq t_A(x\gamma y)$ and $1 - f_A(x) \geq 1 - f_A(x\gamma y)$ or $t_A(y) \geq t_A(x\gamma y)$ and $1 - f_A(y) \geq 1 - f_A(x\gamma y)$. Consequently, $V_A(x) \geq V_A(x\gamma y)$ or $V_A(y) \geq V_A(x\gamma y)$. Therefore, A is a vague weakly completely prime ideal of S. □

Theorem 4.4. *Let T be a non-empty subset of a Γ-semigroup S and δ_T a vague characteristic set of T. Then the following conditions are equivalent:*

(1) *T is a crisp prime ideal of S,*

(2) *δ_T is a vague weakly completely prime ideal of S.*

Proof. Let T be a prime ideal of S and δ_T be the vague characteristic set of T. Since $T \neq \phi$, so δ_T is non-empty. Let $x, y \in S$. Suppose that $x\Gamma y \nsubseteq T$. Then, $t_{\delta_T}(x\gamma y) = 1$ and $f_{\delta_T}(x\gamma y) = 0$ for $\gamma \in \Gamma$. Since T is a prime ideal of S, so $x \in T$ or $y \in T$ which implies that $t_{\delta_T}(x) = t_{\delta_T}(y) = 1$ and $f_{\delta_T}(x) = f_{\delta_T}(y) = 0$. Hence, $t_{\delta_T}(x) \geq t_{\delta_T}(x\gamma y)$ and $1 - f_{\delta_T}(x) \geq 1 - f_{\delta_T}(x\gamma y)$ or $t_{\delta_T}(y) \geq t_{\delta_T}(x\gamma y)$ and $1 - f_{\delta_T}(y) \geq 1 - f_{\delta_T}(x\gamma y)$. Suppose that $x\Gamma y \nsubseteq T$. Then, $t_{\delta_T}(x\gamma y) = 0$ and $f_{\delta_T}(x\gamma y) = 1$ for $\gamma \in \Gamma$. Since T is a crisp prime ideal of S, so $x \notin T$ or $y \notin T$ which implies that $t_{\delta_T}(x) = t_{\delta_T}(y) = 0$ and $f_{\delta_T}(x) = f_{\delta_T}(y) = 1$. Thus, $t_{\delta_T}(x) \geq t_{\delta_T}(x\gamma y)$ and $1 - f_{\delta_T}(x) \geq 1 - f_{\delta_T}(x\gamma y)$ or $t_{\delta_T}(y) \geq t_{\delta_T}(x\gamma y)$ and $1 - f_{\delta_T}(y) \geq 1 - f_{\delta_T}(x\gamma y)$. Therefore, δ_T is a vague weakly completely prime ideal of S.

Conversely, let δ_T be a vague weakly completely prime ideal of S. Then, δ_T is a vague ideal of S. Hence, T is an crisp ideal of $S(cf.$ Theorem III.3[18]). Let $x, y \in S$ be such that $x\Gamma y \subseteq T$. Then $t_{\delta_T}(x\gamma y) = 1$ and $f_{\delta_T}(x\gamma y) = 0$. Let if possible, $x \notin T$ and $y \notin T$. Then, $t_{\delta_T}(x) = t_{\delta_T}(y) = 0$ and $f_{\delta_T}(x) = f_{\delta_T}(y) = 1$. which implies that $t_{\delta_T}(x) < t_{\delta_T}(x\gamma y)$ and $1 - f_{\delta_T}(x) < 1 - f_{\delta_T}(x\gamma y)$ or $t_{\delta_T}(y) < t_{\delta_T}(x\gamma y)$ and $1 - f_{\delta_T}(y) < 1 - f_{\delta_T}(x\gamma y)$. This contradicts our assumption that δ_T is vague weakly completely prime ideal of S. Hence T is a prime ideal of S. □

Remark 4.1. *Theorems 4.2-4.4 hold in the case of semigroup, too.*

5. Corresponding vague weakly completely prime ideal

Definition 5.1. [5] *Let S be a Γ-semigroup. Let us define a relation ρ on $S \times \Gamma$ as follows: $(x, \alpha)\rho(y, \beta)$ if and only if $x\alpha s = y\beta s$ for all $s \in S$ and $\gamma x\alpha = \gamma y\beta$ for all $\gamma \in \Gamma$. Then ρ is an equivalence relation. Let $[x, \alpha]$ denote the equivalence class containing (x, α). Let $L = \{[x, \alpha] : x \in S, \alpha \in \Gamma\}$. Then L is a semigroup with respect to the multiplication defined by $[x, \alpha][y, \beta] = [x\alpha y, \beta]$. This semigroup L is called the left operator semigroup of the Γ-semigroup S. Dually the right operator semigroup R of Γ-semigroup S is defined where the multiplication is defined by $[\alpha, a][\beta, b] = [\alpha a\beta, b]$.*

Unless otherwise stated, throughout this section S denotes a Γ-semigroup and L, R its left and right operator semigroups, respectively.

Definition 5.2. [18] *For a vague set A of R we define a vague set A^* of S by*

$$(V_A)^*(a) = \inf_{\gamma \in \Gamma}\{V_A([\gamma, a])\},$$

where $a \in S$. For a vague set B of S we define a vague set $B^{*'}$ of R by

$$(V_B)^{*'}([\alpha, a]) = \inf_{s \in S}\{V_B(s\alpha a)\},$$

where $[\alpha, a] \in R$. For a vague set C of L, we define a vague set C^+ of S by

$$(V_C)^+(a) = \inf_{\gamma \in \Gamma}\{V_C([a, \gamma])\},$$

where $a \in S$. For a vague set D of S we define a vague set $D^{+'}$ of L by

$$(V_D)^{+'}([a, \alpha]) = \inf_{s \in S}\{V_D(a\alpha s)\},$$

where $[a, \alpha] \in L$.

Now, we recall the following results from [6].

- If P is a prime ideal of L then P^+ is a prime ideal of S.
- If Q is a prime ideal of S then $Q^{+'}$ is a prime ideal of L.
- If P is a prime ideal of R then P^* is a prime ideal of S.
- If Q is a prime ideal of S then $Q^{*'}$ is a prime ideal of R.

Note that for a Γ-semigroup S and its left, right operator semigroups L, R, respectively, four mappings namely $()^+, ()^{+'}, ()^*, ()^{*'}$ occur. They are defined as follows:

- for $I \subseteq R,\ I^* = \{s \in S, [\alpha, s] \in I,\ \forall \alpha \in \Gamma\}$,
- for $P \subseteq S,\ P^{*'} = \{[\alpha, x] \in R : s\alpha x \in P,\ \forall s \in S\}$,
- for $J \subseteq L,\ J^+ = \{s \in S, [s, \alpha] \in J,\ \forall \alpha \in \Gamma\}$,
- for $Q \in S,\ Q^{+'} = \{[x, \alpha] \in L : x\alpha s \in Q,\ \forall s \in S\}$.

Proposition 5.1. [18] *If A be a vague subset of R (the right operator semigroup of a Γ-semigroup S), then $[A_{(\alpha,\beta)}]^* = [A^*]_{(\alpha,\beta)}$, for all $\alpha, \beta \in [0, 1]$ such that the sets are non-empty.*

Proposition 5.2. [18] *Let A be a vague subset of a Γ-semigroup S. Then $[A_{(\alpha,\beta)}]^{*'} = [A^{*'}]_{(\alpha,\beta)}$, for all $\alpha, \beta \in [0, 1]$ such that the sets under consideration are non-empty.*

Proposition 5.3. *If A is a vague weakly completely prime ideal of R, then $1 - A^*$ is a vague subsemigroup of S.*

Proof. Let A be a vague weakly completely prime ideal of R. Then $A_{(\alpha,\beta)}$ is a prime ideal of R. Hence, $[A_{(\alpha,\beta)}]^*$ is a prime ideal of S. Since $[A_{(\alpha,\beta)}]^*$ and $[A^*]_{(\alpha,\beta)}$ are non-empty, so we have $[A_{(\alpha,\beta)}]^* = [A^*]_{(\alpha,\beta)}$. Hence $[A^*]_{(\alpha,\beta)}$ is a prime ideal of S. Consequently, A^* is a vague weakly completely prime ideal of S. Therefore, $1 - A^*$ is a vague subsemigroup of S. □

Proposition 5.4. *If B is a vague weakly completely prime ideal of S, then $1 - B^{*'}$ is a vague subsemigroup of R.*

Proof. Let B be a vague weakly completely prime ideal of S. Then $B_{(\alpha,\beta)}$ is a prime ideal of S. Hence, $[B_{(\alpha,\beta)}]^{*'}$ is a prime ideal of R. Since $[B_{(\alpha,\beta)}]^{*'}$ and $[B^{*'}]_{(\alpha,\beta)}$ are non-empty, so we obtain $[B_{(\alpha,\beta)}]^{*'} = [B^{*'}]_{(\alpha,\beta)}$. Hence, $[B^{*'}]_{(\alpha,\beta)}$ is a prime ideal of R. Consequently, $B^{*'}$ is a vague weakly completely prime ideal of R. Hence $1 - B^{*'}$ is a vague subsemigroup of R. □

Remark 5.1. *The left operator analogous of Propositions 5.3 and 5.4 are true, too.*

Theorem 5.1. *Let S be a Γ-semigroup and R be its right operator semigroup. Then there exists an inclusion preserving bijection $A \mapsto A^{*'}$ between the set of all vague weakly completely prime ideal of R and the set of all vague weakly completely prime ideals of S, where A is a vague weakly completely prime ideal of R.*

Proof. Let $x \in S$. Then

$$(t_A^{*'})^*(x) = \inf_{\alpha \in \Gamma} \left\{ t_A^{*'}([\alpha, x]) \right\} = \inf_{\alpha \in \Gamma} \left\{ \inf_{s \in S} \{ t_A(s\alpha x) \} \right\} \geq t_A(x)$$

and

$$1 - (f_A^{*'})^*(x) = 1 - \inf_{\alpha \in \Gamma} \left\{ f_A^{*'}([\alpha, x]) \right\} = 1 - \inf_{\alpha \in \Gamma} \left\{ \inf_{s \in S} \{ f_A(s\alpha x) \} \right\} \geq 1 - f_A(x),$$

since A is a vague ideal. Consequently, $V_A \subseteq (V_A^{*'})^*$. Again, for $x \in S$, we have

$$(t_A^{*'})^*(x) = \inf_{\alpha\in\Gamma}\left\{t_A^{*'}([\alpha, x])\right\} = \inf_{\alpha\in\Gamma}\left\{\inf_{s\in S}\{t_A(s\alpha x)\}\right\} \leq t_A(x)$$

and

$$1 - (f_A^{*'})^*(x) = 1 - \inf_{\alpha\in\Gamma}\{f_A^{*'}([\alpha, x])\} = 1 - \inf_{\alpha\in\Gamma}\left\{\inf_{s\in S}\{f_A(s\alpha x)\}\right\} \leq 1 - f_A(x),$$

since A is a vague weakly completely prime ideal. Consequently, $V_A \supseteq (V_A^{*'})^*$. Hence $V_A = (V_A^{*'})^*$ and consequently the mapping is one-one. Now, for $[\alpha, x] \in R$,

$$\begin{aligned}(t_A^*)^{*'}([\alpha, x]) = \inf_{s\in S}\{t_A^*(s\alpha x)\} \quad &= \inf_{s\in S}\left\{\inf_{\beta\in\Gamma}\{t_A([\beta, s\alpha x])\}\right\} = \\ &= \inf_{s\in S}\left\{\inf_{\beta\in\Gamma}\{t_A([\beta, s][\alpha, x])\}\right\} \geq \\ &\geq t_A([\alpha, x])\end{aligned}$$

and

$$\begin{aligned}1 - (f_A^*)^{*'}([\alpha, x]) \quad &= 1 - \inf_{s\in S}\{f_A^*(s\alpha x)\} = \\ &= 1 - \inf_{s\in S}\left\{\inf_{\beta\in\Gamma}\{f_A([\beta, s\alpha x])\}\right\} = \\ &= 1 - \inf_{s\in S}\left\{\inf_{\beta\in\Gamma}\{f_A([\beta, s][\alpha, x])\}\right\} \geq \\ &\geq 1 - f_A([\alpha, x]).\end{aligned}$$

Consequently, $V_A \subseteq (V_A^*)^{*'}$. Again, since A is a vague weakly completely prime ideal, so we have $t_A([\beta, s][\alpha, x]) \leq t_A([\beta, s])$ and $1 - f_A([\beta, s][\alpha, x]) \leq 1 - f_A([\beta, s])$, or, $t_A([\beta, s][\alpha, x]) \leq t_A([\alpha, x])$ and $1 - f_A([\beta, s][\alpha, x]) \leq 1 - f_A([\alpha, x])$ for all $s \in S$ and $\beta \in \Gamma$. Hence, for $s = x$ and $\beta = \alpha$, we have $t_A([\beta, s][\alpha, x]) \leq t_A([\alpha, x])$ and $1 - f_A([\beta, s][\alpha, x]) \leq 1 - f_A([\alpha, x])$. Thus,

$$(t_A^*)^{*'}([\alpha, x]) = \inf_{s\in S}\left\{\inf_{\beta\in\Gamma}\{t_A([\beta, s][\alpha, x])\}\right\} \leq t_A([\alpha, x])$$

and

$$1 - (f_A^*)^{*'}([\alpha, x]) = 1 - \inf_{s\in S}\left\{\inf_{\beta\in\Gamma}\{f_A([\beta, s][\alpha, x])\}\right\} \leq 1 - f_A([\alpha, x]).$$

Consequently, $(V_A^*)^{*'} \subseteq V_A$, for all $[\alpha, x] \in R$. Hence $(V_A^*)^{*'} = V_A$. This proves that the mapping is onto. Let A_1 and A_2 be two vague ideals of S such that $A_1 \subseteq A_2$. Then, for all $[\alpha, x] \in R$,

$$t_{A_1}^{*'}([\alpha, x]) = \inf_{s\in S}\{t_{A_1}(s\alpha x)\} \leq \inf_{s\in S}\{t_{A_2}(s\alpha x)\} = t_{A_2}^{*'}([\alpha, x])$$

and

$$1 - f_{A_1}^{*'}([\alpha, x]) = 1 - \inf_{s\in S}\{f_{A_1}(s\alpha x)\} \leq 1 - \inf_{s\in S}\{f_{A_2}(s\alpha x)\} = 1 - f_{A_2}^{*'}([\alpha, x]).$$

Consequently, $A_1^{*'} \subseteq A_2^{*'}$. Similarly, we can show that if $A_1 \subseteq A_2$, where A_1 and A_2 are vague ideals of R, then $A_1^* \subseteq A_2^*$. Therefore, $A \mapsto A^{*'}$ is an inclusion preserving bijection. □

In view of Theorem 4.1 and Theorem 5.1, we have the following theorem.

Theorem 5.2. *Let S be a Γ-semigroup and R be its right operator semigroup. Then there exists an inclusion preserving bijection $1 - A \mapsto 1 - A^{*'}$ between the set of all vague subsemigroups of R and the set of all vague subsemigroups of S, where $1 - A$ is a vague subsemigroup of R.*

Remark 5.2. *Similar result holds for the Γ-semigroup S and the left operator semigroup L of S.*

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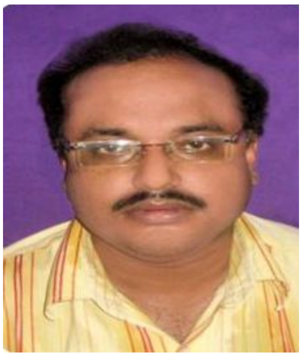

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